UFSCARF-TH-03-14

Integrable $SU(N)$ vertex models with general toroidal boundary conditions

G.A.P. Ribeiro, M.J. Martins and W. Galleas

Universidade Federal de São Carlos

Departamento de Física

C.P. 676, 13565-905 São Carlos(SP), Brasil

Abstract

We formulate the algebraic Bethe ansatz solution of the $SU(N)$ vertex models with rather general non-diagonal toroidal boundary conditions. The reference states needed in the Bethe ansatz construction are found by performing gauge transformations on the Boltzmann weights in the manner of Baxter [1]. The structure of the transfer matrix eigenvectors consists of multi-particle states over such pseudovacuums and the corresponding eigenvalues depend crucially on the boundary matrix eigenvalues. We also discuss for $N = 2$ the peculiar case of twisted boundaries associated to singular matrices.

PACS numbers: 05.50+q, 02.30.IK

Keywords: Algebraic Bethe Ansatz, Lattice Models

July 2003

1 Introduction

The study of vertex models have led to important developments in the field of exactly solvable models in two dimensions [1]. Their transfer matrices are in general constructed from local Boltzmann weights $\mathcal{L}_{\mathcal{A}i}(\lambda)$ where λ denotes a spectral parameter. This operator can be viewed as a matrix on the space of states $\mathcal{A}$ representing, for instance, the horizontal degrees of freedom of the vertex model on the square lattice. Its matrix elements are operators on $\prod_{i=1}^{L} \otimes V_i$ where V_i represents the space of states of the vertical degrees of freedom at each site i of a chain of size L. The corresponding transfer matrix can be expressed in terms of an ordered product of $\mathcal{L}_{\mathcal{A}i}(\lambda)$ operator over the space $\mathcal{A}$ denominated monodromy operator $\mathcal{T}_{\mathcal{A}}(\lambda)$ [2],

$$\mathcal{T}_{\mathcal{A}}(\lambda) = \mathcal{L}_{\mathcal{A}L}(\lambda)\mathcal{L}_{\mathcal{A}L-1}(\lambda)\dots\mathcal{L}_{\mathcal{A}1}(\lambda) \tag{1}$$

In terms of the monodromy matrix, a sufficient condition for integrability is the Yang-Baxter algebra [2, 3] which reads

$$R(\lambda,\mu)\mathcal{T}_{\mathcal{A}}(\lambda) \otimes \mathcal{T}_{\mathcal{A}}(\mu) = \mathcal{T}_{\mathcal{A}}(\mu) \otimes \mathcal{T}_{\mathcal{A}}(\lambda)R(\lambda,\mu) \tag{2}$$

where $R(\lambda,\mu)$ is an invertible matrix over complex numbers acting on the tensor product $\mathcal{A}\otimes\mathcal{A}$ space.

The Yang-Baxter algebra is invariant by the transformation $\mathcal{T}_{\mathcal{A}}(\lambda) \to \mathcal{G}_{\mathcal{A}}\mathcal{T}_{\mathcal{A}}(\lambda)$ provided that the group of c-numbers matrices $\mathcal{G}_{\mathcal{A}}$ satisfies the following property [4]

$$[R(\lambda,\mu), \mathcal{G}_{\mathcal{A}} \otimes \mathcal{G}_{\mathcal{A}}] = 0 \tag{3}$$

An immediate consequence of this symmetry is the possibility to define the operator

$$T(\lambda) = Tr_{\mathcal{A}}[\mathcal{G}_{\mathcal{A}}\mathcal{T}_{\mathcal{A}}(\lambda)] \tag{4}$$

which gives origin to generalized families of commuting transfer matrices.

When the matrix $\mathcal{G}_{\mathcal{A}}$ is non singular a quantum spin chain can be associated with the transfer matrix (4). For sake of simplicity, consider the usual situation in which the spaces $\mathcal{A}$ and V_i are isomorphic and that the $\mathcal{L}_{\mathcal{A}i}(\lambda)$ is proportional to the exchange operator $P_{\mathcal{A}i}$ at

certain special point say $\lambda = 0$. The corresponding one-dimensional Hamiltonian is obtained as a logarithmic derivative of the transfer matrix at point $\lambda = 0$, which reads [4, 11]

$$\mathcal{H} = \sum_{i=1}^{L-1} P_{ii+1} \frac{d\mathcal{L}_{ii+1}(\lambda)}{d\lambda} \mid_{\lambda=0} + \mathcal{G}_L^{-1} P_{L1} \frac{d\mathcal{L}_{L1}(\lambda)}{d\lambda} \mid_{\lambda=0} \mathcal{G}_L \tag{5}$$

Clearly, the admissible $\mathcal{G}_\mathcal{A}$ matrices play the role of more general toroidal boundary conditions than the particular periodic case when $\mathcal{G}_\mathcal{A}$ is the identity matrix, the simplest possibility satisfying relation (3). From the point of view of a vertex model, such general twisted boundary conditions correspond to the introduction of a line of defects along the infinite direction on the cylinder. Though boundary conditions are not expected to influence the infinite volume properties it can change the finite-size corrections of massless systems in a strip of width L which contains fundamental informations concerning the underlying conformal field theories [5]. For instance, in statistical mechanics boundary conditions provide the means to relate the critical behaviour of a variety of different lattice systems such as the Heisenberg spin chain, the Ashkin-Teller and the Potts models [6]. In this sense, it is highly desirable to study integrable models with as much general boundary conditions as possible.

If the boundary matrix $\mathcal{G}_\mathcal{A}$ is diagonal the corresponding transfer matrix (4) can be diagonalized with very little difference from the periodic case because it does not change in a drastic way the properties of the monodromy matrix elements. The same does not occur when $\mathcal{G}_\mathcal{A}$ is non-diagonal, starting from the fact that the reference state of the periodic case, essential to implement Bethe ansatz approaches, is a priori no longer of utility due to the breaking of the original bulk symmetry by the boundary terms. In fact, progress towards solving commuting transfer matrices with general twists by Bethe ansatz techniques are modest as compared with the literature known for the periodic case, specially for solvable vertex models based on Lie algebras, e.g. refs.[7, 8, 9]. To our knowledge, the six vertex model and its higher spin descendants [10, 11] are the only solvable vertex systems analyzed so far with non-diagonal boundary conditions. Even in these cases, the functional relation method used in refs.[10, 11] gives the transfer matrix eigenvalues but not information on the corresponding eigenvectors. The latter is certainly an important step in the program of solving integrable systems.

The purpose of this paper is the formulation of the quantum inverse scattering method for the simplest multistate generalization of the six vertex model having N independent degrees of freedom on each lattice bond. This turns out to be the isotropic $SU(N)$ vertex model whose origin goes back to the work by Uimin [12] and Sutherland [13] on generalized integrable Heisenberg chains with higher symmetry. Its corresponding $\mathcal{L}_{\mathcal{A}i}(\lambda)$ operators can be written as [14, 15]

$$\mathcal{L}_{\mathcal{A}i}(\lambda) = \lambda I_{\mathcal{A}i} + P_{\mathcal{A}i} \tag{6}$$

where as usual $I_{\mathcal{A}i}$ is the identity matrix on the space $\mathcal{A} \otimes V_i$.

The interesting feature of this system is that the admissible symmetries constitute of arbitrary $N \times N$ $\mathcal{G}_{\mathcal{A}}$ matrices due to the standard property $P_{12}A_1 \otimes B_2 = B_2 \otimes A_1 P_{12}$. Therefore this provides us a rich variety of possible diagonal and non-diagonal boundary conditions. In next section, we present the details of the solution of the eigenvalue problem for the transfer matrix in the simplest $N = 2$ case. Interesting enough, we find that the Bethe ansatz solution depends on the eigenvalue problem related to the boundary $\mathcal{G}_{\mathcal{A}}$ matrix. In section 3 we generalize these results for arbitrary values of N by using the nested Bethe ansatz approach. Our conclusions are presented in section 4 as well as a discussion of singular boundaries for the model $N = 2$. In Appendix A we discuss briefly the completeness of the Hilbert space for $N = 2$ and finite L.

2 Algebraic Bethe ansatz for Heisenberg model

The purpose of this section is to determine the eigenvalues and the eigenvectors of the following transfer matrix

$$T(\lambda) = Tr_{\mathcal{A}}[\mathcal{G}_{\mathcal{A}} \mathcal{L}_{\mathcal{A}L}(\lambda) \dots \mathcal{L}_{\mathcal{A}1}(\lambda)] \tag{7}$$

The operator $\mathcal{L}_{\mathcal{A}i}(\lambda)$ is the elementary Boltzmann weights of the isotropic six vertex model which can be written as

$$\mathcal{L}_{\mathcal{A}i}(\lambda) = \begin{pmatrix} \frac{[a(\lambda)+b(\lambda)]}{2} I_i + \frac{[a(\lambda)-b(\lambda)]}{2} \sigma_i^z & \sigma_i^- \\ \sigma_i^+ & \frac{[a(\lambda)+b(\lambda)]}{2} I_i - \frac{[a(\lambda)-b(\lambda)]}{2} \sigma_i^z \end{pmatrix} \tag{8}$$

where $\sigma^{\pm}_{\mathcal{A},i}$ and $\sigma^{z}_{\mathcal{A},i}$ are Pauli matrices acting on the vertical space of states and the weights are $a(\lambda) = \lambda + 1$ and $b(\lambda) = \lambda$. The boundary matrix $\mathcal{G}_{\mathcal{A}}$ is an arbitrary 2×2 matrix over the complex numbers whose matrix elements are denoted by

$$\mathcal{G}_{\mathcal{A}} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{9}$$

An essential ingredient of the quantum inverse scattering is the existence of a reference state such that the action of the monodromy operator in this state gives as a result a triangular matrix. Though each of the operators $\mathcal{L}_{\mathcal{A}i}(\lambda)$ when acting on the trivial spin up $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_i$ or spin down $\begin{pmatrix} 0 \\ 1 \end{pmatrix}_i$ states becomes triangular, such property is not extended to the total monodromy because the off diagonal elements of $\mathcal{G}_{\mathcal{A}}$ are in general non-null. Therefore the standard ferromagnetic pseudovacuum is not useful when both g_{12} and g_{21} are different from zero. In order to find an appropriate reference state we have to introduce a set of gauge transformations similar to that used by Baxter [1] in the solution of the eight vertex model. We replace the local operators $\mathcal{L}_{\mathcal{A}i}(\lambda)$ by new matrices $\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda)$ such that [2]

$$\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda) = M^{-1}_{j+1}\mathcal{L}_{\mathcal{A}j}(\lambda)M_j \tag{10}$$

where M_j are arbitrary invertible 2×2 c-number matrices acting on the space $\mathcal{A}$. After performing this gauge transformations the transfer matrix (7) becomes

$$T(\lambda) = Tr_{\mathcal{A}}[M_1^{-1}\mathcal{G}_{\mathcal{A}}M_{L+1}\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)] \tag{11}$$

where $\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda) = \widetilde{\mathcal{L}}_{\mathcal{A}L}(\lambda)\dots\widetilde{\mathcal{L}}_{\mathcal{A}1}(\lambda)$.

The next step is to look for gauge transformations M_j such that $\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda)$ is annihilated for instance by its lower left element for arbitrary values of the spectral parameter. Representing the matrices M_j by

$$M_j = \begin{pmatrix} x_j & r_j \\ y_j & s_j \end{pmatrix} \tag{12}$$

we can conclude [1] that such annihilation property occurs when the ratio $\frac{x_j}{y_j}$ is a constant for $j = 1, \ldots, L+1$. As a consequence of that we can choose the local reference state $|0\rangle_j$ as

$$|0\rangle_j = \begin{pmatrix} \frac{x_j}{y_j} \\ 1 \end{pmatrix}_j \tag{13}$$

following that the action of the operator $\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda)$ in this state is given by

$$\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda)\,|0\rangle_j = \begin{pmatrix} a(\lambda)\frac{y_j}{y_{j+1}}\,|0\rangle_j & \# \\ 0 & b(\lambda)\frac{y_{j+1}}{y_j}\frac{det[M_j]}{det[M_{j+1}]}\,|0\rangle_j \end{pmatrix} \tag{14}$$

where the symbol # represents general non-null values.

The remaining freedom that we have on the matrix elements of M_j is now used to choose matrices M_1 and M_{L+1} in such way that they transform the boundary matrix $\mathcal{G}_{\mathcal{A}}$ into a diagonal matrix. More precisely, by imposing that

$$M_1^{-1}\mathcal{G}_{\mathcal{A}}M_{L+1} = \begin{pmatrix} \widetilde{g}_1 & 0 \\ 0 & \widetilde{g}_2 \end{pmatrix} \tag{15}$$

it follows that the constrains for the first column elements are

$$\begin{aligned} g_{11}x_{L+1} + g_{12}y_{L+1} &= \widetilde{g}_1 x_1 \\ g_{21}x_{L+1} + g_{22}y_{L+1} &= \widetilde{g}_1 y_1 \end{aligned} \tag{16}$$

while for the second column elements we have

$$\begin{aligned} g_{11}r_{L+1} + g_{12}s_{L+1} &= \widetilde{g}_2 r_1 \\ g_{21}r_{L+1} + g_{22}s_{L+1} &= \widetilde{g}_2 s_1 \end{aligned} \tag{17}$$

At this point we emphasize our assumption that we are dealing with a non-singular boundary matrix. While we have an enormous freedom to choose the second column elements the same does not occur for the first ones because the ratio $\frac{x_j}{y_j}$ needs to be kept fixed to preserve triangularity of $\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda)$. This latter fact together with relation (16) impose a restriction to this

ratio which is precisely the same satisfied by ratio of the components of the eigenvectors of the boundary matrix $\mathcal{G}_{\mathcal{A}}$. Therefore, we have two possibilities for the ratio $p^{(\pm)} = \frac{x_j}{y_j}$ which are

$$p^{(\pm)} = \frac{(g_{11} - g_{22}) \pm \sqrt{(g_{11} - g_{22})^2 + 4 g_{12} g_{21}}}{2 g_{21}} \tag{18}$$

Putting now all these informations together it is possible to build up two appropriate global reference states $|0\rangle^{(\pm)}$ by the tensor product

$$|0\rangle^{(\pm)} = \prod_{j=1}^{L} \otimes \begin{pmatrix} p^{(\pm)} \\ 1 \end{pmatrix}_j \tag{19}$$

At this point the state (19) preserves at least the desirable triangular property of the total monodromy $M_1^{-1} \mathcal{G}_{\mathcal{A}} M_{L+1} \widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)$. Below we shall see that they are indeed eigenstates of the transfer matrix (7) independent of further choices of the elements of the gauge matrices M_j. Further progress is made by recasting the Yang-Baxter algebra for the gauge transformed monodromy $\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)$ in the form of commutation relations for the creation and annihilation fields. In order to do that it is convenient to represent $\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)$ by the following 2×2 matrix

$$\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix} \tag{20}$$

As a consequence of the triangular property (14) we are able to derive important relations for the diagonal elements of the transformed monodromy matrix

$$\begin{aligned} \widetilde{A}(\lambda)\, |0\rangle^{(\pm)} &= [a(\lambda)]^L \frac{y_1}{y_{L+1}}\, |0\rangle^{(\pm)} \\ \widetilde{D}(\lambda)\, |0\rangle^{(\pm)} &= [b(\lambda)]^L \frac{y_{L+1}}{y_1} \frac{det[M_1]}{det[M_{L+1}]}\, |0\rangle^{(\pm)} \end{aligned} \tag{21}$$

besides the annihilation property

$$\widetilde{C}(\lambda)\, |0\rangle^{(\pm)} = 0 \tag{22}$$

Now taking into account that gauge matrices M_j are themselves symmetries allowed by the property (3) it is not difficult to show that the gauge transformed monodromy $\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)$ matrix

satisfies the same Yang-Baxter algebra as the original monodromy matrix $\mathcal{T}_{\mathcal{A}}(\lambda)$. In other words, we have that $\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)$ satisfies the relation

$$R(\lambda,\mu)\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)\otimes\widetilde{\mathcal{T}}_{\mathcal{A}}(\mu)=\widetilde{\mathcal{T}}_{\mathcal{A}}(\mu)\otimes\widetilde{\mathcal{T}}_{\mathcal{A}}(\lambda)R(\lambda,\mu) \tag{23}$$

where in our case the R-matrix is given by

$$R(\lambda,\mu)=\begin{pmatrix} a(\lambda-\mu) & 0 & 0 & 0 \\ 0 & 1 & b(\lambda-\mu) & 0 \\ 0 & b(\lambda-\mu) & 1 & 0 \\ 0 & 0 & 0 & a(\lambda-\mu) \end{pmatrix} \tag{24}$$

This means that we have the same set of commutation rules of the periodic six vertex model [2, 3] however now for the gauged matrix elements. Out of sixteen possible relations three of them are of great use, namely

$$\widetilde{A}(\lambda)\widetilde{B}(\mu) = \frac{a(\mu-\lambda)}{b(\mu-\lambda)}\widetilde{B}(\mu)\widetilde{A}(\lambda)-\frac{1}{b(\mu-\lambda)}\widetilde{B}(\lambda)\widetilde{A}(\mu) \tag{25}$$

$$\widetilde{D}(\lambda)\widetilde{B}(\mu) = \frac{a(\lambda-\mu)}{b(\lambda-\mu)}\widetilde{B}(\mu)\widetilde{D}(\lambda)-\frac{1}{b(\lambda-\mu)}\widetilde{B}(\lambda)\widetilde{D}(\mu) \tag{26}$$

$$\left[\widetilde{B}(\lambda),\widetilde{B}(\mu)\right] = 0 \tag{27}$$

The fields $\widetilde{B}(\lambda)$ are then interpreted as a kind of creation operators over the pseudovacuum $|0\rangle^{(\pm)}$ and a natural ansatz for the eigenvectors $|\phi\rangle^{(\pm)}$ of the transfer matrix $T(\lambda)$ is

$$|\phi\rangle^{(\pm)}=\prod_{j=1}^{n_{\pm}}\widetilde{B}(\lambda_j^{(\pm)})\,|0\rangle^{(\pm)} \tag{28}$$

The eigenvalue problem $T(\lambda)\,|\phi\rangle^{(\pm)}=\Lambda^{(\pm)}(\lambda)\,|\phi\rangle^{(\pm)}$ now becomes

$$\left[\widetilde{g}_1\widetilde{A}(\lambda)+\widetilde{g}_2\widetilde{D}(\lambda)\right]|\phi\rangle^{(\pm)}=\Lambda^{(\pm)}(\lambda)\,|\phi\rangle^{(\pm)} \tag{29}$$

and it can be solved in the same way as the periodic six vertex model [2], i.e by taking the fields $\widetilde{A}(\lambda)$ and $\widetilde{D}(\lambda)$ through the creation operators $\widetilde{B}(\lambda)$ with the help of the commutation rules (25-27). One peculiarity here, however, is that the calculations involving the action of the diagonal fields $\widetilde{A}(\lambda)$ and $\widetilde{D}(\lambda)$ over the reference state $|0\rangle^{(\pm)}$ requires extra simplifications

to eliminate unnecessary dependence of the gauge matrices elements. They are carried out by using the help of Eqs.(16 - 18) and our final result for the eigenvalues $\Lambda^{(\pm)}(\lambda)$ are

$$\Lambda^{(\pm)}(\lambda) = g^{(\pm)}[a(\lambda)]^L \prod_{i=1}^{n_\pm} \frac{\lambda_i^{(\pm)} - \lambda + \frac{1}{2}}{\lambda_i^{(\pm)} - \lambda - \frac{1}{2}} + g^{(\mp)}[b(\lambda)]^L \prod_{i=1}^{n_\pm} \frac{\lambda - \lambda_i^{(\pm)} + \frac{3}{2}}{\lambda - \lambda_i^{(\pm)} + \frac{1}{2}} \tag{30}$$

provided that the rapidities $\lambda_i^{(\pm)}$ satisfy the following Bethe ansatz equations

$$\left[\frac{\lambda_i^{(\pm)} + \frac{1}{2}}{\lambda_i^{(\pm)} - \frac{1}{2}}\right]^L = \frac{g^{(\mp)}}{g^{(\pm)}} \prod_{\substack{j=1 \\ j\neq i}}^{n_\pm} \frac{\lambda_i^{(\pm)} - \lambda_j^{(\pm)} + 1}{\lambda_i^{(\pm)} - \lambda_j^{(\pm)} - 1} \tag{31}$$

where we have performed the convenient shift $\lambda_i^{(\pm)} \rightarrow \lambda_i^{(\pm)} - \frac{1}{2}$. The phase factors $g^{(\pm)}$ are just the eigenvalues of the matrix $\mathcal{G}_{\mathcal{A}}$

$$g^{(\pm)} = \frac{(g_{11} + g_{22}) \pm \sqrt{(g_{11} - g_{22})^2 + 4g_{12}g_{21}}}{2} \tag{32}$$

Rather remarkably, we see that the final form of the eigenvalues and Bethe ansatz equations resemble much that of the isotropic six vertex model with diagonal boundary if we replace the diagonal twists by the eigenvalues of the non-diagonal boundary $\mathcal{G}_{\mathcal{A}}$ matrix. The eigenvectors can also be thought as multi-particle states in which the integers $n_\pm \leq L$ play the role of particle number sectors. We emphasize, however, that the corresponding basic creation fields are much more sophisticated operators than that of the periodic six vertex model [2]. It is tempting to think that the two possible ways we have at our disposal to build up the Hilbert space is related to the remaining Z_2 symmetry allowed by boundary terms. One expects therefore that it should be possible to obtain the eigenvalues of the transfer matrix either from the $|0\rangle^{(+)}$ or $|0\rangle^{(-)}$ pseudovacuums. Indeed, we have verified this fact by numerically solving the equations for some values of L and comparing them to exact diagonalization of the transfer matrix (7). We note, however, that a given eigenvalue of the transfer matrix is in general obtained at different particle sectors $n_\pm$ over the $|0\rangle^{(\pm)}$ reference states. For example, the eigenvalue $g^+[a(\lambda)]^L + g^-[b(\lambda)]^L$ can be obtained either from the zero-particle state $|0\rangle^{(+)}$ or as a L-particle state over the pseudovacuum $|0\rangle^{(-)}$. In Appendix A, we present details of our study for $L = 2$ in which Eqs.(30-31) can be solved by analytical means. Our numerical results

up to $L = 4$ suggest that two possible branches of the Bethe ansatz solutions (31) produce the complete spectrum of the transfer matrix (7). It would be interesting to further investigate the completeness of the Bethe ansatz (31) by adapting the recent arguments developed by Baxter [16] to the case of non-diagonal twists.

We now can derive similar results for the spin chain that commutes with the transfer matrix (7). The corresponding spin-1/2 XXX Hamiltonian follows from expression (5) and it is given by

$$\mathcal{H} = \mathcal{J} \sum_{j=1}^{L} \left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{\sigma_j^z \sigma_{j+1}^z}{2} \right) \tag{33}$$

with the following boundary condition

$$\begin{pmatrix} \sigma_{L+1}^+ \\ \sigma_{L+1}^- \\ \sigma_{L+1}^z \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} g_{11}^2 & -g_{21}^2 & -g_{11}g_{21} \\ -g_{12}^2 & g_{22}^2 & g_{12}g_{22} \\ -2g_{11}g_{12} & 2g_{21}g_{22} & g_{11}g_{22} + g_{12}g_{21} \end{pmatrix} \begin{pmatrix} \sigma_1^+ \\ \sigma_1^- \\ \sigma_1^z \end{pmatrix} \tag{34}$$

Its eigenvalues $E^{(\pm)} = \frac{dLog[\Lambda^{(\pm)}(\lambda)]}{d\lambda} |_{\lambda=0}$ are

$$E^{(\pm)} = \mathcal{J} \sum_{j=1}^{n_\pm} \frac{1}{\lambda_j^{\pm 2} - \frac{1}{4}} + \frac{\mathcal{J} L}{2} \tag{35}$$

where $\lambda_i^{(\pm)}$ satisfy the Bethe ansatz equations (31).

Our final comment concerns with the comparison between our results (30-31) and that of refs.[10, 11] in the isotropic limit case when the trigonometric weights become rational functions. We see that they are in accordance for the common non-diagonal boundary $\mathcal{G}_\mathcal{A} = \begin{pmatrix} 0 & g_{12} \\ g_{21} & 0 \end{pmatrix}$ apart from the fact that our numbers of roots $n_\pm$ can vary up to L while that of refs.[10, 11] are fixed at L. This implies that for non-diagonal boundary conditions the complete solution of the isotropic limit does not follows directly from that found for the anisotropic six vertex model [10, 11]. This means that even for this particular non-diagonal boundary the results (30-31) are novel in the literature.

3 Nested Bethe ansatz for $SU(N)$ model

The purpose of this section is to generalize the results of the previous section for general N. We wish to diagonalize the transfer matrix (7), where now the operator $\mathcal{L}_{\mathcal{A}i}(\lambda)$ is

$$\mathcal{L}_{\mathcal{A}i}(\lambda) = a(\lambda) \sum_{\alpha=1}^{N} \hat{e}_{\alpha\alpha}^{(\mathcal{A})} \otimes \hat{e}_{\alpha\alpha}^{(i)} + b(\lambda) \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \hat{e}_{\alpha\alpha}^{(\mathcal{A})} \otimes \hat{e}_{\beta\beta}^{(i)} + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \hat{e}_{\alpha\beta}^{(\mathcal{A})} \otimes \hat{e}_{\beta\alpha}^{(i)} \tag{36}$$

where $\hat{e}_{ij}^{(V)}$ are the standard Weyl matrices whose elements acting on the space V are $[\hat{e}_{ij}^{(V)}]_{kl} = \delta_{ik}\delta_{jl}$. In this basis the boundary matrix $\mathcal{G}_{\mathcal{A}}$ is generally represented by

$$\mathcal{G}_{\mathcal{A}} = \sum_{\alpha,\beta=1}^{N} g_{\alpha\beta} \hat{e}_{\alpha\beta}^{(\mathcal{A})} \tag{37}$$

As before we have to seek for suitable references states by imposing the gauge transformation (10) for each operator (36) and require that they are up triangular when acting on such pseudovacuum. Denoting the gauge matrices by $M_j = \sum_{\alpha,\beta=1}^{N} m_j(\alpha,\beta)\hat{e}_{\alpha\beta}^{(\mathcal{A})}$ we find that such triangular property is fully achieved when the following ratios relations are satisfied

$$p_{\alpha,\beta} = \frac{m_j(\alpha,\beta)}{m_j(N,\beta)} = \frac{m_{j+1}(\alpha,\beta)}{m_{j+1}(N,\beta)}, \quad \alpha,\beta = 1,\dots,N-1 \tag{38}$$

for each $j = 1,\dots,L+1$. In terms of these ratios the local reference state $|0\rangle_j$ assume the form

$$|0\rangle_j = \begin{pmatrix} p_{1,1} \\ p_{2,1} \\ \vdots \\ p_{N-1,1} \\ 1 \end{pmatrix}_j \tag{39}$$

Other important ingredient is the action of the gauge transformed operator $\widetilde{\mathcal{L}}_{\mathcal{A}j}(\lambda)$ over the local state of reference. This now can be represented by the following $N \times N$ matrix on

the space $\mathcal{A}$

$$\widetilde{\mathcal{L}}_{\mathcal{A}j}\left|0\right\rangle_j = \begin{pmatrix} a(\lambda)\frac{f_1^j}{f_1^{j+1}}\left|0\right\rangle_j & \# & \# & \cdots & \# \\ 0 & b(\lambda)\frac{f_2^j}{f_2^{j+1}}\left|0\right\rangle_j & \# & \cdots & \# \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b(\lambda)\frac{f_N^j}{f_N^{j+1}}\left|0\right\rangle_j \end{pmatrix}_{N\times N} \tag{40}$$

where the variables f_α^j are given by

$$f_\alpha^j = \begin{cases} m_j(N,\alpha), & \alpha = 1,\dots,N-1 \\ \left(\prod_{i=1}^{N-1}\frac{1}{m_j(N,i)}\right) det[M_j], & \alpha = N \end{cases} \tag{41}$$

Similarly to the previous section we can take advantage of the remaining freedom of the elements of the gauge matrices to transform $M_1^{-1}\mathcal{G}_{\mathcal{A}}M_{L+1}$ into a diagonal matrix. By imposing this condition the matrices elements of M_1 and M_{L+1} become related by the expression

$$\sum_{\gamma=1}^{N} g_{\alpha\gamma}m_{L+1}(\gamma,\beta) = m_1(\alpha,\beta)\widetilde{g}_\beta, \quad \alpha,\beta = 1,\dots,N \tag{42}$$

where $\widetilde{g}_\alpha$ represent the diagonal elements of the transformed boundary matrix.

Equations (38) and (42) together impose constrains to the possible values ratios $p_{\alpha,\beta}$ which turns out to be same conditions satisfied by the ratio of the components of the eigenvectors of boundary matrix $\mathcal{G}_{\mathcal{A}}$. This means that we have N possible choices for $p_{\alpha,1}^{(l)}$ $l = 1,\dots,N$ and consequently from Eq.(39) N kind of suitable local references states $\left|0\right\rangle_j^{(l)}$. A natural ansatz for the N possible choices of global reference states are

$$\left|0\right\rangle^{(l)} = \prod_{j=1}^{L}\otimes\left|0\right\rangle_j^{(l)} \quad l = 1,\dots,N \tag{43}$$

The next step is to write a suitable representation for the gauge transformed monodromy matrix in the auxiliary space $\mathcal{A}$. The triangular property (40) suggests us to seek for the structure used in nested Bethe ansatz diagonalization of the periodic $SU(N)$ vertex models

[14, 15] which is

$$\widetilde{\mathcal{T}}(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}_1(\lambda) & \cdots & \widetilde{B}_{N-1}(\lambda) \\ \widetilde{C}_1(\lambda) & \widetilde{D}_{11}(\lambda) & \cdots & \widetilde{D}_{1N-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{C}_{N-1}(\lambda) & \widetilde{D}_{N-11}(\lambda) & \cdots & \widetilde{D}_{N-1N-1}(\lambda) \end{pmatrix}_{N\times N} \tag{44}$$

The triangularity property (40) implies that the fields $\widetilde{B}_i(\lambda)$ play the role of creations operators, $\widetilde{C}_i(\lambda)$ are annihilation fields while the diagonal operator $\widetilde{A}(\lambda)$ and $\widetilde{D}_{ii}(\lambda)$ acts on the reference state $|0\rangle^{(l)}$ as

$$\widetilde{A}(\lambda)\,|0\rangle^{(l)} = [a(\lambda)]^L \frac{f_1^1}{f_1^{L+1}}\,|0\rangle^{(l)} \tag{45}$$

$$\widetilde{D}_{ii}(\lambda)\,|0\rangle^{(l)} = [b(\lambda)]^L \frac{f_{i+1}^1}{f_{i+1}^{L+1}}\,|0\rangle^{(l)},\ i = 1, \ldots, N-1 \tag{46}$$

To construct other eigenvectors we shall use the commutation relations between the gauge transformed monodromy matrix elements. The arguments used in section 2 allows to conclude that these commutation rules are the same as that already known for the periodic $SU(N)$ models [14, 15]. The most useful relations for subsequent derivations are

$$\widetilde{A}(\lambda)\widetilde{B}_i(\mu) = \frac{a(\mu-\lambda)}{b(\mu-\lambda)}\widetilde{B}_i(\mu)\widetilde{A}(\lambda) - \frac{1}{b(\mu-\lambda)}\widetilde{B}_i(\lambda)\widetilde{A}(\mu) \tag{47}$$

$$\widetilde{D}_{ij}(\lambda)\widetilde{B}_k(\mu) = \frac{1}{b(\lambda-\mu)}\widetilde{B}_p(\mu)\widetilde{D}_{iq}(\lambda)r^{(1)}(\lambda-\mu)_{pq}^{jk} - \frac{1}{b(\lambda-\mu)}\widetilde{B}_j(\lambda)\widetilde{D}_{ik}(\mu) \tag{48}$$

$$\widetilde{B}_i(\lambda)\widetilde{B}_j(\mu) = \widetilde{B}_p(\mu)\widetilde{B}_q(\lambda)r^{(1)}(\lambda-\mu)_{pq}^{ij} \tag{49}$$

where $r^{(1)}(\lambda)_{pq}^{ij}$ are the elements of the R-matrix associated to the $SU(N-1)$ vertex model.

In terms of the gauge transformed fields, the eigenvalue problem for the transfer matrix $T(\lambda)$ becomes

$$\left[\widetilde{g}_1\widetilde{A}(\lambda) + \sum_{i=1}^{N-1}\widetilde{g}_{i+1}\widetilde{D}_{ii}(\lambda)\right]|\phi\rangle^{(l)} = \Lambda^{(l)}(\lambda)\,|\phi\rangle^{(l)} \tag{50}$$

where $|\phi\rangle^{(l)}$ denotes the eigenvectors. Previous experience with these models [14, 15] suggests us to suppose that eigenvectors can be written in terms of the following linear combination

$$|\phi\rangle^{(l)} = \widetilde{B}_{a_1}(\lambda_1^{(1,l)})\ldots\widetilde{B}_{a_{m_1^l}}(\lambda_{m_1^l}^{(1,l)})\mathcal{F}^{a_{m_1^l}\ldots a_1}\,|0\rangle^{(l)} \tag{51}$$

where sum over repeated indices $a_n = 1, \dots N-1$ is assumed. At this stage the components $\mathcal{F}^{a_{m_1^l} \dots a_1}$ are thought as coefficients of an arbitrary linear combination that are going to be determined a posteriori.

By carrying on the fields $\widetilde{A}(\lambda)$ and $\widetilde{D}_{ii}(\lambda)$ over the multi-particle state (51) we generate terms that are proportional to $|\phi\rangle^{(l)}$ and those that are not the so-called unwanted terms. The first ones will contribute directly to the eigenvalue $\Lambda^{(l)}(\lambda)$ and are obtained by keeping only the first term of the commutation rules (47-48). These calculations are by now standard in the literature and here we present only the main results of the action of the transfer matrix on the eigenvector $|\phi\rangle^{(l)}$ which is

$$\begin{aligned} T(\lambda)\,|\phi\rangle^{(l)} &= \widetilde{g}_1 \frac{f_1^1}{f_1^{L+1}} [a(\lambda)]^L \prod_{j=1}^{m_1^l} \frac{a(\lambda_j^{(1,l)} - \lambda)}{b(\lambda_j^{(1,l)} - \lambda)} \,|\phi\rangle^{(l)} \\ &+ [b(\lambda)]^L \prod_{j=1}^{m_1^l} \frac{1}{b(\lambda - \lambda_j^{(1,l)})} \widetilde{B}_{b_1}(\lambda_1^{(1,l)}) \dots \widetilde{B}_{b_{m_1^l}}(\lambda_{m_1^l}^{(1,l)}) T^{(1)}(\lambda, \{\lambda_i^{(1,l)}\})_{b_1 \dots b_{m_1^l}}^{a_1 \dots a_{m_1^l}} \mathcal{F}^{a_{m_1^l} \dots a_1} \,|0\rangle^{(l)} \\ &+ \text{unwanted terms} \end{aligned} \tag{52}$$

All the pieces entering the above expression can be summarized as follows. The terms $T^{(1)}(\lambda, \{\lambda_i^{(1,l)}\})_{b_1 \dots b_{m_1^l}}^{a_1 \dots a_{m_1^l}}$ are transfer matrix elements of an auxiliary inhomogeneous problem related to the $SU(N-1)$ vertex model with twisted boundaries $\widetilde{\mathcal{G}}$ defined by

$$T^{(1)}(\lambda, \{\lambda_i^{(1,l)}\})_{b_1 \dots b_{m_1^l}}^{a_1 \dots a_{m_1^l}} = r^{(1)}(\lambda - \lambda_1^{(1,l)})_{b_1 d_1}^{a a_1} r^{(1)}(\lambda - \lambda_2^{(1,l)})_{b_2 d_2}^{d_1 a_2} \dots r^{(1)}(\lambda - \lambda_{m_1^l}^{(1,l)})_{b_{m_1^l} d_{m_1^l}}^{d_{m_1^l - 1} a_{m_1^l}} \widetilde{\mathcal{G}}_{a d_{m_1^l}} \tag{53}$$

where $\widetilde{\mathcal{G}}_{ab}$ denotes the elements of the boundary matrix $\widetilde{\mathcal{G}}$ given by

$$\widetilde{\mathcal{G}} = \begin{pmatrix} \widetilde{g}_2 \frac{f_2^1}{f_2^{L+1}} & \# & \# & \cdots & \# \\ 0 & \widetilde{g}_3 \frac{f_3^1}{f_3^{L+1}} & \# & \cdots & \# \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \widetilde{g}_N \frac{f_N^1}{f_N^{L+1}} \end{pmatrix}_{N-1 \times N-1} \tag{54}$$

The unwanted terms are originated when the variables $\lambda_i^{(1,l)}$ of the multi-particle state (51) are exchanged with the spectral parameter λ due to the second part of the commutation rules (47-48). It is possible to collect these terms in closed forms, thanks to the commutation rule

(49) which makes possible to relate different ordered multi-particle states. It turns out that all the unwanted terms are canceled out provided that the rapidities $\lambda_i^{(1,l)}$ satisfy the following restriction,

$$\widetilde{g}_1 \frac{f_1^1}{f_1^{L+1}} \left[\frac{a(\lambda_i^{(1,l)})}{b(\lambda_i^{(1,l)})}\right]^L \prod_{\substack{j=1 \\ j\neq i}}^{m_1^l} b(\lambda_i^{(1,l)} - \lambda_j^{(1,l)}) \frac{a(\lambda_j^{(1,l)} - \lambda_i^{(1,l)})}{b(\lambda_j^{(1,l)} - \lambda_i^{(1,l)})} \mathcal{F}^{a_{m_1^l} \dots a_1} =$$
$$T^{(1)}(\lambda = \lambda_i^{(1,l)}, \{\lambda_j^{(1,l)}\})_{a_1 \cdots a_{m_1^l}}^{b_1 \cdots b_{m_1^l}} \mathcal{F}^{b_{m_1^l} \dots b_1}, i = 1, \dots, m_1^l \tag{55}$$

Now we reached a point which is fundamental to diagonalize $T^{(1)}(\lambda, \{\lambda_i^{(1,l)}\})$ in order to compute the eigenvalues of $T(\lambda)$ and at the same time to solve Eq.(55). This becomes possible if we require that $\mathcal{F}^{a_{m_1^l} \dots a_1}$ is an eigenvector of the auxiliary transfer matrix with eigenvalue $\Lambda^{(1)}(\lambda, \{\lambda_i^{(1,l)}\})$, namely

$$T^{(1)}(\lambda, \{\lambda_i^{(1,l)}\})_{a_1 \cdots a_{m_1^l}}^{b_1 \cdots b_{m_1^l}} \mathcal{F}^{b_{m_1^l} \dots b_1} = \Lambda^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \mathcal{F}^{a_{m_1^l} \dots a_1} \tag{56}$$

Inspection of Eq.(52) and Eq.(55) together with Eq.(56) shows that the eigenvalue of $T(\lambda)$ is

$$\Lambda(\lambda)^{(l)} = \widetilde{g}_1 \frac{f_1^1}{f_1^{L+1}} [a(\lambda)]^L \prod_{i=1}^{m_1^l} \frac{a(\lambda_i^{(1,l)} - \lambda)}{b(\lambda_i^{(1,l)} - \lambda)} + [b(\lambda)]^L \prod_{i=1}^{m_1^l} \frac{1}{b(\lambda - \lambda_i^{(1,l)})} \Lambda^{(1)}(\lambda, \{\lambda_i^{(1,l)}\}) \tag{57}$$

and the nested Bethe ansatz equations (55) become

$$\widetilde{g}_1 \frac{f_1^1}{f_1^{L+1}} \left[\frac{a(\lambda_i^{(1,l)})}{b(\lambda_i^{(1,l)})}\right]^L \prod_{\substack{j=1 \\ j\neq i}}^{m_1^l} b(\lambda_i^{(1,l)} - \lambda_j^{(1,l)}) \frac{a(\lambda_j^{(1,l)} - \lambda_i^{(1,l)})}{b(\lambda_j^{(1,l)} - \lambda_i^{(1,l)})} = \Lambda^{(1)}(\lambda = \lambda_i^{(1,l)}, \{\lambda_j^{(1,l)}\}), \quad i = 1, \dots, m_1^l \tag{58}$$

In order to solve the eigenvalue problem (56) it is necessary to introduce a second algebraic Bethe ansatz for the eigenvectors $\mathcal{F}^{a_{m_1^l} \dots a_1}$. Because the boundary matrix $\widetilde{\mathcal{G}}$ is triangular there is no need to perform gauge transformations to find an appropriate reference state for $T^{(1)}(\lambda, \{\lambda_i^{(1,l)}\})$. We can use, for instance, the usual ferromagnetic pseudovacuum build up by tensor product of elementary $(N-1)$-dimensional $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{N-1}$ vectors. As a result the solution

(56) becomes very similar to that of the periodic $SU(N-1)$ vertex model in the presence of inhomogeneities. Since this problem has been extensively discussed in the literature we will only present our final results for the main eigenvalue problem (50). It turns out that the eigenvalues of the transfer matrix $\Lambda^{(l)}(\lambda)$ is given by

$$\begin{aligned}\Lambda^{(l)}(\lambda;\{\lambda_i^{(1,l)}\},\dots,\{\lambda_i^{(N-1,l)}\}) \;=\; & \widetilde{g}_1\frac{f_1^1}{f_1^{L+1}}[a(\lambda)]^L\prod_{j=1}^{m_1^l}\frac{a(\lambda_j^{(1,l)}-\lambda)}{b(\lambda_j^{(1,l)}-\lambda)} \\ & +[b(\lambda)]^L\sum_{k=1}^{N-2}\widetilde{g}_{k+1}\frac{f_{k+1}^1}{f_{k+1}^{L+1}}\prod_{j=1}^{m_k^l}\frac{a(\lambda-\lambda_j^{(k,l)})}{b(\lambda-\lambda_j^{(k,l)})}\prod_{j=1}^{m_{k+1}^l}\frac{a(\lambda_j^{(k+1,l)}-\lambda)}{b(\lambda_j^{(k+1,l)}-\lambda)} \\ & +[b(\lambda)]^L\widetilde{g}_N\frac{f_N^1}{f_N^{L+1}}\prod_{j=1}^{m_{N-1}^l}\frac{a(\lambda-\lambda_j^{(N-1,l)})}{b(\lambda-\lambda_j^{(N-1,l)})} \qquad (59)\end{aligned}$$

The rapidities $\{\lambda_i^{(k,l)}\}$ $k=1,\dots,N$ parameterize the multi-particle states of the nesting problem at step k and are required to satisfy the following nested Bethe ansatz equations

$$\frac{\widetilde{g}_1}{\widetilde{g}_2}\frac{f_1^1f_2^{L+1}}{f_1^{L+1}f_2^1}\left[\frac{a(\lambda_i^{(1,l)})}{b(\lambda_i^{(1,l)})}\right]^L=\prod_{\substack{j=1\\ j\neq i}}^{m_1^l}-\frac{a(\lambda_i^{(1,l)}-\lambda_j^{(1,l)})}{a(\lambda_j^{(1,l)}-\lambda_i^{(1,l)})}\prod_{\substack{j=1\\ j\neq i}}^{m_2^l}\frac{a(\lambda_j^{(2,l)}-\lambda_i^{(1,l)})}{b(\lambda_j^{(2,l)}-\lambda_i^{(1,l)})} \qquad (60)$$

$$\frac{\widetilde{g}_k}{\widetilde{g}_{k+1}}\frac{f_k^1f_{k+1}^{L+1}}{f_k^{L+1}f_{k+1}^1}\prod_{j=1}^{m_{k-1}^l}\frac{a(\lambda_i^{(k,l)}-\lambda_j^{(k-1,l)})}{b(\lambda_i^{(k,l)}-\lambda_j^{(k-1,l)})}=\prod_{\substack{j=1\\ j\neq i}}^{m_k^l}-\frac{a(\lambda_i^{(k,l)}-\lambda_j^{(k,l)})}{a(\lambda_j^{(k,l)}-\lambda_i^{(k,l)})}\prod_{\substack{j=1\\ j\neq i}}^{m_{k+1}^l}\frac{a(\lambda_j^{(k+1,l)}-\lambda_i^{(k,l)})}{b(\lambda_j^{(k+1,l)}-\lambda_i^{(k,l)})}, \qquad (61)$$

$$k=2,\dots,N-2$$

$$\frac{\widetilde{g}_{N-1}}{\widetilde{g}_N}\frac{f_{N-1}^1f_N^{L+1}}{f_{N-1}^{L+1}f_N^1}\prod_{\substack{j=1\\ j\neq i}}^{m_{N-2}^l}\frac{a(\lambda_i^{(N-1,l)}-\lambda_j^{(N-2,l)})}{b(\lambda_i^{(N-1,l)}-\lambda_j^{(N-2,l)})}=\prod_{\substack{j=1\\ j\neq i}}^{m_{N-1}^l}-\frac{a(\lambda_i^{(N-1,l)}-\lambda_j^{(N-1,l)})}{a(\lambda_j^{(N-1,l)}-\lambda_i^{(N-1,l)})} \qquad (62)$$

The final step is to carry out simplifications on the phase factors $\widetilde{g}_i\frac{f_i^1}{f_i^{L+1}}$ with the help of the constraints (38) and (42). After a cumbersome algebra it is possible to show that such factors are just the eigenvalues $g^{(i)}$ of the boundary matrix $\mathcal{G}_{\mathcal{A}}$. To make sure that the different possibilities we have at hand for the ratios $p_{\alpha,\beta}^{(l)}$ do not lead us to singular gauge matrices we choose to order them for each l-th choice of pseudovacuum by $g^{(l)}=g^{(l+N)}$ for $l=1,\dots,N$. Taking into account this ordering as well as performing the shifts $\{\lambda_j^{(k,l)}\}\rightarrow\{\lambda_j^{(k,l)}\}-k/2$ our

result (59) for the eigenvalue becomes

$$\Lambda^{(l)}(\lambda;\{\lambda_i^{(1,l)}\},\ldots,\{\lambda_i^{(N-1,l)}\}) = g^{(l)}[a(\lambda)]^L \prod_{j=1}^{m_1^l} \frac{\lambda_j^{(1,l)}-\lambda+\frac{1}{2}}{\lambda_j^{(1,l)}-\lambda-\frac{1}{2}}$$
$$+[b(\lambda)]^L \sum_{k=1}^{N-2} g^{(l+k)} \prod_{j=1}^{m_k^l} \frac{\lambda-\lambda_j^{(k,l)}+\frac{k+2}{2}}{\lambda-\lambda_j^{(k,l)}+\frac{k}{2}} \prod_{j=1}^{m_{k+1}^l} \frac{\lambda_j^{(k+1,l)}-\lambda+\frac{1-k}{2}}{\lambda_j^{(k+1,l)}-\lambda-\frac{k+1}{2}}$$
$$+[b(\lambda)]^L g^{(l+N-1)} \prod_{j=1}^{m_{N-1}^l} \frac{\lambda-\lambda_j^{(N-1,l)}+\frac{N+1}{2}}{\lambda-\lambda_j^{(N-1,l)}+\frac{N-1}{2}} \tag{63}$$

and the nested Bethe ansatz equations can be compactly written as

$$\left[\frac{\lambda_i^{(a,l)}+\frac{\delta_{a,1}}{2}}{\lambda_i^{(a,l)}-\frac{\delta_{a,1}}{2}}\right]^L = \frac{g^{(l+a)}}{g^{(l+a-1)}} \prod_{b=1}^{N-1} \prod_{\substack{k=1 \\ k\neq i}}^{m_b^l} \frac{\lambda_i^{(a,l)}-\lambda_k^{(b,l)}+\frac{C_{a,b}}{2}}{\lambda_i^{(a,l)}-\lambda_k^{(b,l)}-\frac{C_{a,b}}{2}}, \quad i=1,\ldots,m_a^l; \quad a=1,\ldots,N-1 \tag{64}$$

where C_{ab} is the Cartan matrix elements of the $SU(N)$ Lie algebra.

We see that the results for the eigenvalues and the Bethe ansatz equations is similar to that expected from the $SU(N)$ vertex model with diagonal twists giving by the eigenvalues of the boundary matrix $\mathcal{G}_{\mathcal{A}}$. It remains to be investigated whether this interesting feature is particular of the $SU(N)$ symmetry or it also works in other isotropic vertex models such as those invariant by the $O(N)$ and $Sp(2N)$ Lie algebras.

4 Concluding remarks

In this paper we have been able to apply the quantum inverse scattering program to solve exactly the isotropic $SU(N)$ vertex model with non-diagonal twisted boundary conditions. We find that the eigenvectors can be constructed in terms of multi-particle states over N possible pseudovacuums. The Bethe ansatz results for the eigenvalues are similar to that of the $SU(N)$ model with diagonal boundaries in which the eigenvalues of the boundary matrix $\mathcal{G}_{\mathcal{A}}$ play the role of the diagonal twists.

We expect that our results can be generalized without further difficulties to accommodate the solution of vertex models based on the $SL(N|M)$ super Lie algebra [13, 17, 18] with

general non-diagonal twists. These will include interesting systems of correlated electrons on a lattice such as the one-dimensional supersymmetric t-J model [19] and the so-called Essler, Korepin and Schoutens superconducting model [20] with arbitrary symmetry breaking boundary conditions. With more effort we hope that our approach can be further generalized to include the trigonometric deformation of those vertex models based on the $U_q[SL(N|M)]$ symmetry. In these cases, however, we recall that the possible $\mathcal{G}_{\mathcal{A}}$ matrices compatible with integrability belong to a smaller group formed by one-dimensional dilatations and the discrete Z_{N+M} symmetry.

Other interesting issue that deserves investigation is the situation when the boundary matrix $\mathcal{G}_{\mathcal{A}}$ becomes singular. For example, one would like to ask it is still possible to exhibit eigenvectors of the transfer matrix that are given by direct tensor product of N-dimensional vectors such as the reference states of sections 2 and 3. We have studied this problem in the simplest case $N = 2$ and surprisingly we found a family of such states $|\phi\rangle^{(n)}$ which are

$$|\phi\rangle^{(n)} = \prod_{i=1}^{n} \otimes \begin{pmatrix} -\frac{g_{22}}{g_{21}} \\ 1 \end{pmatrix}_i \prod_{i=n+1}^{L} \otimes \begin{pmatrix} \frac{g_{11}}{g_{21}} \\ 1 \end{pmatrix}_i , \quad n = 0, 1, \ldots, L \tag{65}$$

whose corresponding eigenvalues $\Lambda^{(n)}(\lambda)$ have also the following simple factorized form

$$\Lambda^{(n)}(\lambda) = (g_{11} + g_{22})[a(\lambda)]^{L-n}[b(\lambda)]^n \tag{66}$$

This result prompted us to study further properties of the transfer matrix (7) when $\mathcal{G}_{\mathcal{A}}$ is a singular matrix. Our study for finite L up to six sites reveals that the roots of the characteristic polynomial of $T(\lambda)$ are exactly the eigenvalues (66) whose degeneracy is the binomial coefficient $d_n = \frac{L!}{(L-n)!n!}$. In the case of singular boundary matrix $T(\lambda)$ becomes defective since it has fewer than 2^L independent eigenvectors. To each distinct eigenvalue $\Lambda^{(n)}(\lambda)$ we find only one eigenvector which is precisely the state (65) and therefore the total number of independent states is $L + 1$. These results are strong evidences that $T(\lambda)$ behaves as a non-derogatory matrix and we conjecture that its Jordan decomposition for arbitrary L should be

$$T(\lambda) = diag(J_0, J_1, \cdots, J_L) \tag{67}$$

where J_n is a $d_n \times d_n$ Jordan matrix is given by

$$J_n = \begin{pmatrix} \Lambda^{(n)}(\lambda) & 1 & 0 & \cdots & 0 \\ 0 & \Lambda^{(n)}(\lambda) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda^{(n)}(\lambda) \end{pmatrix}_{d_n \times d_n} \tag{68}$$

This turns out to be a remarkable example how boundary conditions can change in a drastic way the Hilbert space of integrable models. At this point it is natural to ask what happens to the Bethe ansatz states (28) when one gradually varies the boundary matrix towards the singular manifold. In particular, if we can figure out the kind of Bethe states in each sector $n_{\pm}$ that should collapse to the eigenvectors (65). A precise answer to this question as well as possible generalizations of these results for arbitrary N has eluded us so far.

Acknowledgements

We would like to thank the organizers of the Workshop Flux, Charge, Topology and Statistics 2003, Amsterdam, where part of this work was carried out. The authors G.A.P Ribeiro and W. Galleas thank Fapesp (Fundação de Amparo à Pesquisa do Estado de São Paulo) for financial support. The work of M.J. Martins has been partially support by the Brazilian Research Council-CNPq and Fapesp.

Appendix A : Completeness for $L = 2$

This appendix is concerned with the study of the completeness of the Bethe ansatz solution (31) for $L = 2$, i.e. that all four eigenvalues of the transfer matrix are obtained either by starting with $|0\rangle^{(+)}$ or with $|0\rangle^{(-)}$. Let us first begin with $|0\rangle^{(+)}$ whose corresponding eigenvalue $\Lambda_0^{(+)}(\lambda)$ is clearly

$$\Lambda_0^{(+)}(\lambda) = g^{(+)}[a(\lambda)]^2 + g^{(-)}[b(\lambda)]^2 \tag{A.1}$$

The next step is to solve the Bethe ansatz equations for the one-particle state $\widetilde{B}(\lambda_1)\left|0\right\rangle^{(+)}$. As a result we find two possible rapidities given by

$$\lambda_1^{\pm} = -\frac{1}{2}\frac{\left(\sqrt{g^{(+)}} \pm \sqrt{g^{(-)}}\right)}{\left(\sqrt{g^{(+)}} \mp \sqrt{g^{(-)}}\right)} \tag{A.2}$$

giving us the following one-particle $\Lambda_1^{\pm}$ eigenvalues

$$\Lambda_1^{\pm} = a(\lambda)b(\lambda)\left(g^{(+)} + g^{(-)}\right) \pm \sqrt{g^{(+)}g^{(-)}} \tag{A.3}$$

Repeating similar exercise for the two-particle state $\widetilde{B}(\lambda_1)\widetilde{B}(\lambda_2)\left|0\right\rangle^{(+)}$ we have

$$\lambda_{1,2} = \frac{g^{(+)} + g^{(-)} \pm 2I\sqrt{g^{(+)}g^{(-)}}}{2\left(g^{(-)} - g^{(+)}\right)} \tag{A.4}$$

and the corresponding eigenvalue is

$$\Lambda_2(\lambda) = g^{(-)}[a(\lambda)]^2 + g^{(+)}[b(\lambda)]^2 \tag{A.5}$$

An exact diagonalization of the transfer matrix (7) corroborates these four possible eigenvalues for $L = 2$. Note also that (A.5) is exactly the eigenvalue associated to reference state $\left|0\right\rangle^{(-)}$. The others eigenvalues (A.1) and (A.3) are easily obtained from $\left|0\right\rangle^{-}$ by noticing that to each solution $\lambda_i^{(+)}$ one can find the corresponding $\lambda_i^{(-)}$ through the reflection $\lambda_i^{(-)} = -\lambda_i^{(+)}$ symmetry. We have also investigated numerically this problem for $L = 3, 4$ and found that both references states can lead to the complete spectrum of $T(\lambda)$.

References

[1] R.J. Baxter, "Exactly Solved Models in Statistical Mechanics", Academic Press, New York, 1982.

[2] L.A. Takhtajan and L.D. Faddeev, Russ. Math. Sur., 34(1979) 11.

[3] V.E. Korepin, G. Izergin and N.M. Bogoliubov, "Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe ansatz", Cambridge University Press, 1992, Cambridge.

[4] H.J. de Vega, *Nucl. Phys. B, 240(1984) 495*

[5] J.L. Cardy, *Nucl. Phys. B, 275(1986) 200*

[6] F.C. Alcaraz, M.N. Barber and M.T. Batchelor, *Ann. Phys. 182(1988) 280*

[7] N.Yu. Reshitikhin, *Lett. Math. Phys. 14(1987) 235*

[8] A. Kuniba and J. Suzuki, *Commun. Math. Phys. 173(1995) 225*

[9] M.J. Martins and P.B. Ramos, *Nucl. Phys. B, 500(1997) 579*

[10] M.T. Batchelor, R.J. Baxter, M.J.O. Rourke, C.M. Yung, *J. Phys. A: Math. Gen. 28(1995) 2759*

[11] C.M. Yung and M.T. Batchelor, *Nucl. Phys. B, 446 (1995) 461*

[12] V.G. Uimin, *JETP Lett. 12 (1970) 225*

[13] B. Sutherland, *Phys. Rev. B 12 (1975) 3795*

[14] P.P. Kulish and N.Yu. Reshetikhin, *Sov.Phys.JETP 53 (1981) 108*, *J. Phys. A: Math. Gen. 16(1983) L591*

[15] O. Babelon, H.J. de Vega and C.M. Viallet, *Nucl. Phys. B, 200(1982) 266*

[16] R.J. Baxter, *J. Stat. Phys. 108(2002) 1*

[17] C.L. Schultz, Physica A, 122(1983) 71; J.H.H. Perk and C.L. Schultz, Non-linear Integrable systems - Classical and Quantum Theory, Eds. M. Jimbo and T. Miwa, World Scientific, p. 135

[18] P.P. Kulish, *J. Sov. Math. 35 (1986) 2648*

[19] P. Schlottmann, *Phys. Rev. B, 36(1987) 5177*; H.J. de Vega and E. Lopes, *Phys. Rev. Lett. 67(1991) 489*; A. Foerster and M. Korowski, *Nucl. Phys. B, 396 (1993) 611*; F.H.L. Essler and V.E. Korepin, *Phys. Rev. B, 46(1992) 9147*

[20] F.H.L. Essler, V.E.. Korepin and K. Schoutens, *Phys. Rev. Lett., 68(1992) 2960, ibid 70 (1993)73*